CHAPTER 5

Small-Disturbance Flow over Two-Dimensional Airfoils

Chapter 4:

1. The small-disturbance problem for a wing was established.
2. The problem is separated into the solution of two linear sub-problems, namely the thickness and lifting problems.

Chapter 5:

The thickness and lifting problems for airfoil will be solved. These solutions can then be added to yield the complete small-disturbance solution for the flow past a thin airfoil.

5.1 Symmetric Airfoil with Nonzero Thickness at Zero AOA

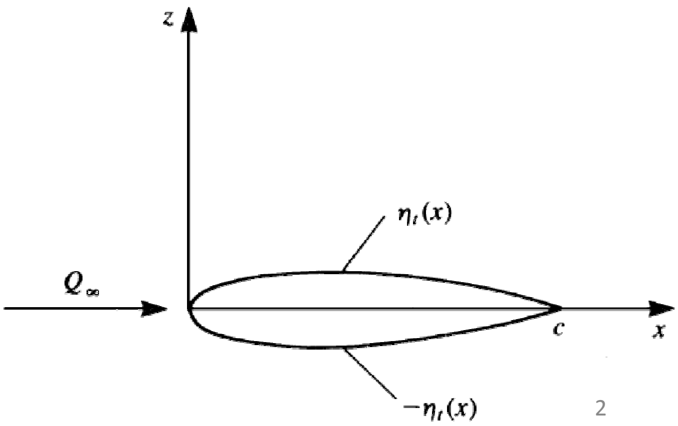
2D symmetric airfoil, with a thickness distribution of η_t (x), at zero angle of attack

$$\begin{cases} \nabla^2 \Phi = 0 & (5.1) \\ \dfrac{\partial \Phi}{\partial z}(x, 0\pm) = \pm \dfrac{d\eta_t}{dx} Q_\infty & (5.2) \end{cases} \longrightarrow \quad w(x, 0\pm) \mp (d\eta_t/dx) Q_\infty = 0$$

Similar to 3D Eq. (4.30)

Recall that:

1. B.C. transferred to the z = 0 plane.
2. B.C. at far from the body is automatically fulfilled by the basic source, doublet, or vortex elements.



5.1 Symmetric Airfoil with Nonzero Thickness at Zero AOA

The symmetry of problem (relative to z = 0 plane) $\longrightarrow$ source distribution

The potential of a single source:

$$\Phi_{\sigma_0} = \frac{\sigma_0}{2\pi} \ln r = \frac{\sigma_0}{2\pi} \ln \sqrt{(x - x_0)^2 + z^2} = \frac{\sigma_0}{4\pi} \ln[(x - x_0)^2 + z^2] \qquad (5.3)$$

The local radial velocity component at an arbitrary point (x, z):

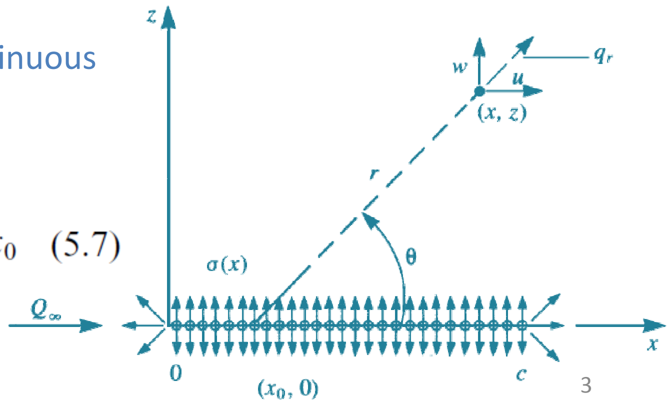
$$q_r = \frac{\sigma_0}{2\pi r} \qquad q_\theta = 0$$

$$\xrightarrow{(u,w) = q_r (\cos\theta, \sin\theta)}$$

$$\begin{cases} u = \dfrac{\partial \Phi_{\sigma_0}}{\partial x} = \dfrac{\sigma_0}{2\pi} \dfrac{x - x_0}{(x - x_0)^2 + z^2} \\[2ex] w = \dfrac{\partial \Phi_{\sigma_0}}{\partial z} = \dfrac{\sigma_0}{2\pi} \dfrac{z}{(x - x_0)^2 + z^2} \end{cases}$$

The airfoil thickness effect is modeled by a continuous σ(x) distribution along the x axis.
The velocity potential:

$$\Phi(x, z) = \frac{1}{2\pi} \int_0^c \sigma(x_0) \ln \sqrt{(x - x_0)^2 + z^2}\, dx_0 \qquad (5.7)$$



5.1 Symmetric Airfoil with Nonzero Thickness at Zero AOA

The velocity field due to source distribution:

$$u(x, z) = \frac{1}{2\pi} \int_0^c \sigma(x_0) \frac{x - x_0}{(x - x_0)^2 + z^2}\, dx_0$$

$$w(x, z) = \frac{1}{2\pi} \int_0^c \sigma(x_0) \frac{z}{(x - x_0)^2 + z^2}\, dx_0$$

Chapter 3 (3.14)

$$w(x, 0\pm) = \lim_{z \to \pm 0} w(x, z) = \pm \frac{\sigma(x)}{2}$$

OR

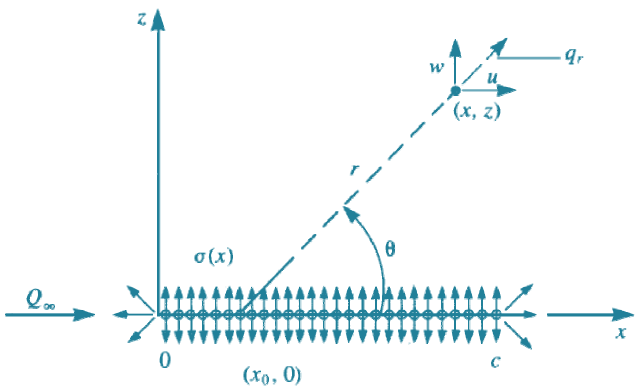
obtaining by observing the volume flow rate

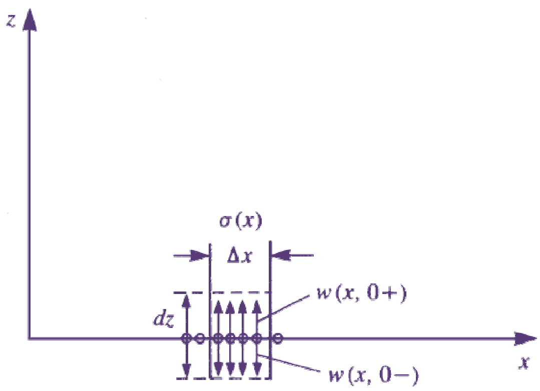
$$\begin{cases} \Sigma = \sigma(x, y)\Delta x \\ dz \to 0 \end{cases}$$

volumetric flow

$$2w(x, 0+)\Delta x = \sigma(x)\Delta x$$

$$w(x, 0\pm) = \pm \frac{\sigma(x)}{2} \qquad (5.10)$$





5.1 Symmetric Airfoil with Nonzero Thickness at Zero AOA

Eq. (5.10) into B.C.

$$\frac{\partial \Phi}{\partial z}(x, 0\pm) = \pm \frac{d\eta_t}{dx} Q_\infty = \pm \frac{\sigma(x)}{2} \longrightarrow \boxed{\sigma(x) = 2Q_\infty \frac{d\eta_t}{dx} \quad (5.11)}$$

The source distribution is easily obtained

$$\Phi(x, z) = \frac{Q_\infty}{\pi} \int_0^c \frac{d\eta_t(x_0)}{dx} \ln \sqrt{(x - x_0)^2 + z^2}\, dx_0 \qquad (5.12)$$

The velocity potential differentiating to obtain the velocity field

$$u(x, z) = \frac{Q_\infty}{\pi} \int_0^c \frac{d\eta_t(x_0)}{dx} \frac{x - x_0}{(x - x_0)^2 + z^2} dx_0 \qquad (5.13)$$

$$w(x, z) = \frac{Q_\infty}{\pi} \int_0^c \frac{d\eta_t(x_0)}{dx} \frac{z}{(x - x_0)^2 + z^2} dx_0 \qquad (5.14)$$

points not lying on the strip ($0 < x < c$, $z = 0$)

The axial velocity component at z = 0

$$u(x, 0) = \frac{Q_\infty}{\pi} \int_0^c \frac{d\eta_t(x_0)}{dx} \frac{1}{(x - x_0)} dx_0 \qquad (5.15)$$

5.1 Symmetric Airfoil with Nonzero Thickness at Zero AOA

Using Bernoulli equation with Small Disturbance Assumptions (*Chapter 4*):

$$p - p_\infty = -\rho Q_\infty \frac{\partial \Phi}{\partial x} = -\rho Q_\infty u(x, 0) \quad (5.16)$$

$$u(x, 0) = \frac{Q_\infty}{\pi} \int_0^c \frac{d\eta_t(x_0)}{dx} \frac{1}{(x - x_0)} dx_0 \qquad (5.15)$$

$$C_p = \frac{p - p_\infty}{(1/2)\rho Q_\infty^2} = -2\frac{u(x, 0)}{Q_\infty}$$

$$C_p = \frac{-2}{\pi} \int_0^c \frac{d\eta_t(x_0)}{dx} \frac{1}{(x - x_0)} dx_0$$

Symmetric u-velocity $\longrightarrow$ Same pressure distribution for upper & lower surface

$$\Delta p = p_l - p_u = 0$$

The aerodynamic lift

$$L = \int_0^c \Delta p\, dx = 0$$

5.1 Symmetric Airfoil with Nonzero Thickness at Zero AOA

The drag force

$$D = \int_0^c p_u \frac{d\eta_t}{dx}\,dx - \int_0^c p_l \frac{-d\eta_t}{dx}\,dx = 2\int_0^c p_u \frac{d\eta_t}{dx}\,dx \qquad (5.21)$$

Eq. (5.15)
Eq. (5.16)

into Eq. (5.21) $\longrightarrow$ $D = -2\rho \frac{Q_\infty^2}{\pi}\int_0^c \int_0^c \frac{[d\eta_t(x_0)/dx][d\eta_t(x)/dx]}{x - x_0}\,dx_0\,dx$

symmetry properties of the integrand

$$D = 0$$

The symmetrical airfoil at zero angle of attack does not generate lift, drag, or pitching moment. Evaluation of the velocity distribution needs to be done only to add this thickness effect to the lifting thin airfoil problem

Calculating axial velocity or pressure on the airfoil:

$$u(x,0) = \frac{Q_\infty}{\pi}\int_0^c \frac{d\eta_t(x_0)}{dx}\frac{1}{(x-x_0)}\,dx_0 \qquad C_p = \frac{-2}{\pi}\int_0^c \frac{d\eta_t(x_0)}{dx}\frac{1}{(x-x_0)}\,dx_0$$

Approaching to $x = x_0$ from left $\longrightarrow$ *The integrand goes to $-\infty$*

Approaching to $x = x_0$ from right $\longrightarrow$ *The integrand goes to $+\infty$*

5.1 Symmetric Airfoil with Nonzero Thickness at Zero AOA

The **Cauchy principal value** of the improper integral

$$\int_a^b f(x_0)\,dx_0 \qquad f(x_0) \to \infty \quad \text{at} \quad x_0 = x \quad \text{and} \quad a < x < b$$

defined by

$$\int_a^b f(x_0)\,dx_0 = \lim_{\epsilon\to 0}\left[\int_a^{x-\epsilon} f(x_0)\,dx_0 + \int_{x+\epsilon}^b f(x_0)\,dx_0\right]$$

As an example:

$$\begin{aligned}\int_0^c \frac{dx_0}{x-x_0} &= \lim_{\epsilon\to 0}\left[\int_0^{x-\epsilon}\frac{dx_0}{x-x_0} - \int_{x+\epsilon}^c \frac{dx_0}{x_0-x}\right]\\ &= \lim_{\epsilon\to 0}\left[-\ln(x-x_0)|_0^{x-\epsilon} - \ln(x_0-x)|_{x+\epsilon}^c\right]\\ &= \lim_{\epsilon\to 0}[-\ln\epsilon + \ln x - \ln(c-x) + \ln\epsilon] = \ln\frac{x}{c-x}\end{aligned}$$

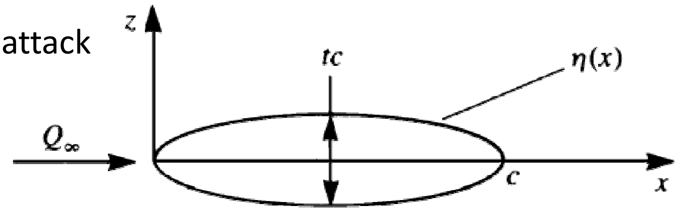
A frequently used principal value integral in many small-disturbance flow applications is the **Glauert integral** which has the form

$$\int_0^\pi \frac{\cos n\theta_0}{\cos\theta_0 - \cos\theta}\,d\theta_0 = \frac{\pi\sin n\theta}{\sin\theta}, \qquad n = 0, 1, 2, \ldots \qquad (5.22)$$

5.1 Symmetric Airfoil with Nonzero Thickness at Zero AOA

Example: *Flow Past an Ellipse*

ellipse with a thickness of $t \cdot c$ at zero angle of attack



$$\frac{[x-(c/2)]^2}{(c/2)^2} + \frac{\eta^2}{(tc/2)^2} = 1$$

Or

$$\eta = \pm t\sqrt{x(c-x)} \quad \longrightarrow \quad \frac{d\eta}{dx} = \pm\frac{t}{2}\frac{c-2x}{\sqrt{x(c-x)}}$$

$$u(x,0) = \frac{Q_\infty}{\pi}\int_0^c \frac{t}{2}\frac{c-2x_0}{\sqrt{x_0(c-x_0)}}\frac{1}{(x-x_0)}\,dx_0$$

This integral needs to be evaluated in terms of its principal value

To enable use of Eq. (5.22) the following transformation is introduced:

$$\begin{cases} x = \dfrac{c}{2}(1-\cos\theta) \\[2ex] dx = \dfrac{c}{2}\sin\theta\, d\theta \end{cases}$$

which transforms the straight chord line into a semicircle. The leading edge of the ellipse (x = 0) is now at θ = 0 and the trailing edge (x = c) is at θ = π.

5.1 Symmetric Airfoil with Nonzero Thickness at Zero AOA

With the aid of this transformation:

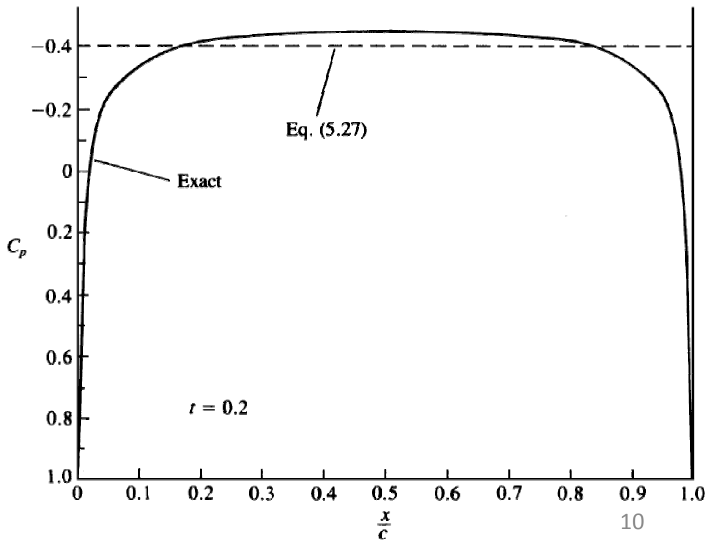
$$\frac{d\eta_t}{dx} = \frac{t}{2}\frac{c-c(1-\cos\theta)}{\sqrt{(c/2)(1-\cos\theta)[c-(c/2)(1-\cos\theta)]}} = t\frac{\cos\theta}{\sin\theta}$$

Substituting this into the u component:

$$u(x,0) = \frac{tQ_\infty}{\pi}\int_0^\pi \frac{\cos\theta_0}{\cos\theta_0 - \cos\theta}\,d\theta_0 \xrightarrow[\text{for } n=1]{\text{Eq. (5.22)}} u(x,0) = tQ_\infty \qquad (5.26)$$

The pressure coefficient:

$$C_p = -2t \qquad (5.27)$$



Solution near the front and rear stagnation points is incorrect.

As the thickness ratio decreases the pressure distribution becomes more flat with a smaller stagnation region and therefore the accuracy of this solution improves.

5.2 Zero-Thickness Airfoil at AOA

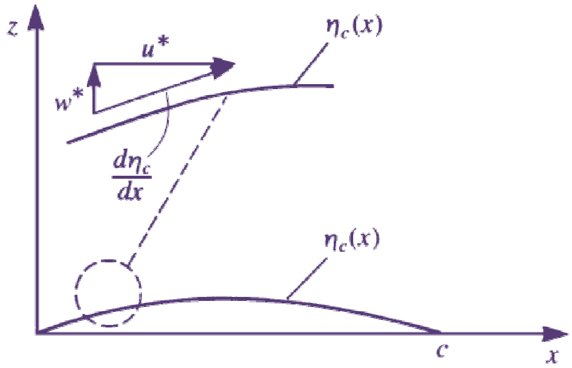
Thin cambered airfoil, at an angle of attack α
The continuity equation & B.C. For small-disturbance inviscid, incompressible, and irrotational transferred to the z = 0 plane

$$\nabla^2 \Phi = 0 \qquad (5.28)$$

$$\frac{\partial \Phi}{\partial z}(x, 0\pm) = Q_\infty \left(\frac{d\eta_c}{dx} \cos\alpha - \sin\alpha \right) \approx Q_\infty \left(\frac{d\eta_c}{dx} - \alpha \right) \qquad (5.29)$$

Thus, the slope of the local (total) velocity must be equal to the camberline slope

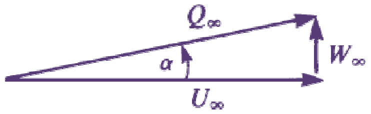
$$\frac{w^*}{u^*} = \frac{\partial \Phi^* / \partial z}{\partial \Phi^* / \partial x} = \frac{d\eta_c}{dx}$$



Point vortex in the *x*–*z* plane, located at a point (x_0, 0) with a strength of γ_0

$$\Phi_{\gamma_0} = -\frac{\gamma_0}{2\pi}\theta = -\frac{\gamma_0}{2\pi} \tan^{-1}\left(\frac{z}{x - x_0} \right) \qquad (5.30)$$

$$q_\theta = -\frac{\gamma_0}{2\pi r}, \qquad q_r = 0$$



5.2 Zero-Thickness Airfoil at AOA

Cartesian coordinates the components of the velocity:

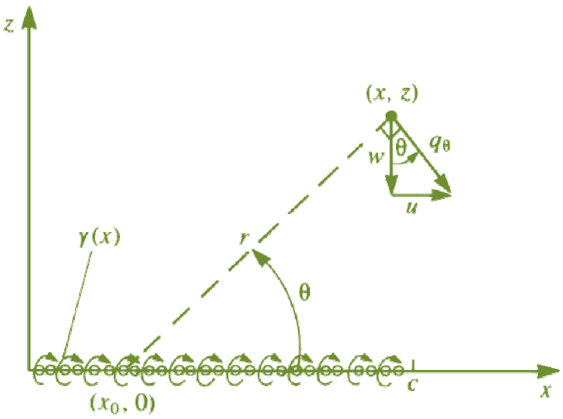
$$(u, w) = q_\theta(\sin\theta, -\cos\theta)$$

OR

Differentiating Eq. (5.30)

$$u = \frac{\partial \Phi_{\gamma_0}}{\partial x} = \frac{\gamma_0}{2\pi} \frac{z}{(x - x_0)^2 + z^2}$$

$$w = \frac{\partial \Phi_{\gamma_0}}{\partial z} = -\frac{\gamma_0}{2\pi} \frac{x - x_0}{(x - x_0)^2 + z^2}$$



Point is placed on the *x* axis

$$w = \frac{-\gamma_0}{2\pi(x - x_0)} \qquad (x \neq x_0)$$

The velocity potential and the resulting velocity field, due to vortex distribution:

$$\Phi(x, z) = \frac{-1}{2\pi} \int_0^c \gamma(x_0) \tan^{-1}\left(\frac{z}{x - x_0} \right) dx_0 \qquad (5.34)$$

$$u(x, z) = \frac{1}{2\pi} \int_0^c \gamma(x_0) \frac{z}{(x - x_0)^2 + z^2} dx_0 \qquad (5.35)$$

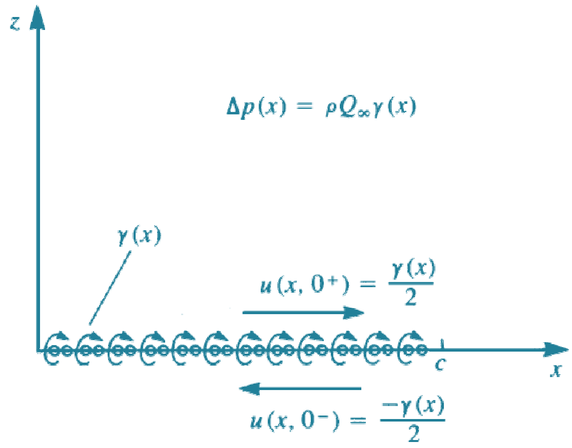
$$w(x, z) = \frac{-1}{2\pi} \int_0^c \gamma(x_0) \frac{x - x_0}{(x - x_0)^2 + z^2} dx_0 \qquad (5.36)$$

5.2 Zero-Thickness Airfoil at AOA

The x component of the velocity above (+) and below (−) a vortex distribution:

$$u(x, 0\pm) = \lim_{z \to \pm 0} u(x, z) = \frac{\pm \gamma(x)}{2}$$

from Section 3.14



The w component of the velocity at z = 0

$$w(x, 0) = \frac{-1}{2\pi} \int_0^c \gamma(x_0) \frac{dx_0}{x - x_0} \qquad (5.38)$$

The unknown vortex distribution γ (x) has to satisfy the zero normal flow boundary condition on the airfoil.

Eq. (5.38) into Eq. (5.29)

$$\frac{\partial \Phi(x, 0)}{\partial z} = w(x, 0) = Q_\infty \left(\frac{d\eta_c}{dx} - \alpha \right)$$

$$\frac{-1}{2\pi} \int_0^c \gamma(x_0) \frac{dx_0}{x - x_0} = Q_\infty \left(\frac{d\eta_c}{dx} - \alpha \right), \quad 0 < x < c \qquad (5.39)$$

5.2 Zero-Thickness Airfoil at AOA

However, the solution to this equation is **not unique** and an additional physical condition is required.

Recall that: The flow leave the trailing edge smoothly and the velocity there be finite, that is

$$\nabla \Phi < \infty \quad \text{(at trailing edges)}$$

$$\gamma(x = c) = 0$$

Kutta condition

The pressure distribution can be calculated by the steady-state Bernoulli equation for small-disturbance flow over the airfoil

$$p - p_\infty = -\rho Q_\infty u(x, 0\pm) = \mp \rho Q_\infty \frac{\gamma}{2}$$

The pressure difference across the airfoil

$$\Delta p = p_l - p_u = p_\infty - \rho Q_\infty \left(-\frac{\gamma}{2} \right) - \left[p_\infty - \rho Q_\infty \left(\frac{\gamma}{2} \right) \right] = \rho Q_\infty \gamma \qquad (5.43)$$

The pressure coefficient with the small-disturbance assumption

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho Q_\infty^2} = \mp \frac{\gamma}{Q_\infty}$$

The pressure difference coefficient between lower and upper surfaces

$$\Delta C_p = 2 \frac{\gamma}{Q_\infty}$$

5.3 Classical Solution of the Lifting Problem

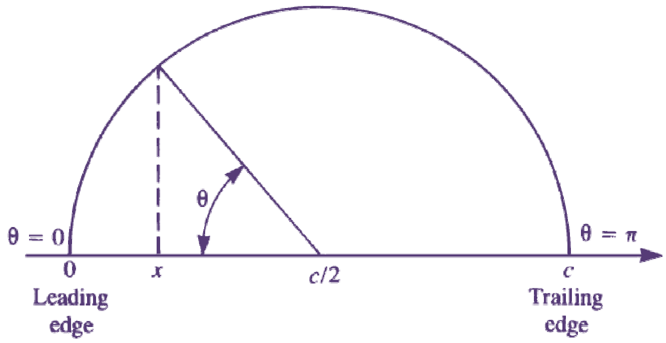
The solution for the aerodynamic loads on the thin, lifting airfoil requires the given γ(x) on the airfoil. This can be obtained by solving the integral equation (5.39).

The classical approach is to approximate γ(x) by a trigonometric expansion and then the problem reduces to finding the coefficient values of this expansion.

$$\frac{-1}{2\pi}\int_0^c \gamma(x_0)\frac{dx_0}{x-x_0} = Q_\infty\left(\frac{d\eta_c}{dx}-\alpha\right), \quad 0<x<c \qquad (5.39)$$

transformation into trigonometric variables

$$\begin{cases} x = \dfrac{c}{2}(1-\cos\theta) \\ dx = \dfrac{c}{2}\sin\theta\, d\theta \end{cases}$$



$$\frac{-1}{2\pi}\int_0^\pi \gamma(\theta_0)\frac{\sin\theta_0\, d\theta_0}{\cos\theta_0-\cos\theta} = Q_\infty\left[\frac{d\eta_c(\theta)}{dx}-\alpha\right], \quad 0<\theta<\pi \qquad (5.46)$$

The transformed Kutta condition now has the form $\gamma(\pi)=0$

$\gamma\ (x)=?$

5.3 Classical Solution of the Lifting Problem

A trigonometric expansion of the form:

$$\sum_{n=1}^{\infty} A_n \sin(n\theta) \longrightarrow$$ *satisfy the Kutta condition*

Experimental evidence shows a large suction peak at the airfoil's leading edge

modeled by

a function whose value is large at the leading edge and reduces to 0 at the trailing edge

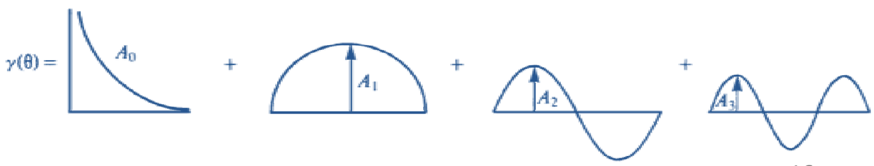
$$A_0\cot\frac{\theta}{2} = A_0\frac{1+\cos\theta}{\sin\theta}$$ *Cotangent function*

The suggested solution for the circulation:

$$\gamma(\theta) = \underbrace{2Q_\infty}\left[A_0\frac{1+\cos\theta}{\sin\theta}+\sum_{n=1}^{\infty}A_n\sin(n\theta)\right] \qquad (5.48)$$

An additional advantage of first term is that it induces a constant downwash on the airfoil, as will be evident later

To cancel the $2Q_\infty$ term on the right-hand side of Eq. (5.46)



5.3 Classical Solution of the Lifting Problem

Determining the values of the A_n constant:

Substituting Eq. (5.48) into B.C. Eq. (5.46)

$$\frac{-1}{2\pi}\int_0^{\pi} 2Q_\infty\left[A_0\frac{1+\cos\theta_0}{\sin\theta_0}+\sum_{n=1}^{\infty}A_n\sin(n\theta_0)\right]\frac{\sin\theta_0\,d\theta_0}{\cos\theta_0-\cos\theta} = Q_\infty\left[\frac{d\eta_c(\theta)}{dx}-\alpha\right] \quad 1$$

Recalling Glauert's integral:

$$\int_0^{\pi}\frac{\cos n\theta_0}{\cos\theta_0-\cos\theta}\,d\theta_0=\frac{\pi\sin n\theta}{\sin\theta}, \qquad n=0,1,2,\dots$$

and replacing 1 by cos (0θ), the first term of the integral becomes

$$\frac{-1}{\pi}A_0\int_0^{\pi}\frac{\cos 0\theta_0+\cos\theta_0}{\cancel{\sin\theta_0}}\frac{\cancel{\sin\theta_0}\,d\theta_0}{\cos\theta_0-\cos\theta}=\frac{-1}{\pi}A_0(0+\pi)=-A_0 \quad 2$$

For the terms with the coefficients A_1, A_2, . . . , the following trigonometric relation is used

$$\sin n\theta_0\sin\theta_0=\frac{1}{2}[\cos(n-1)\theta_0-\cos(n+1)\theta_0], \qquad n=1,2,3,\dots$$

5.3 Classical Solution of the Lifting Problem

This allows the presentation of the nth term in the following form

$$\frac{-1}{\pi}\int_0^{\pi}[A_n\sin(n\theta_0)]\frac{\sin\theta_0\,d\theta_0}{\cos\theta_0-\cos\theta}=\frac{-A_n}{2\pi}\int_0^{\pi}[\cos(n-1)\theta_0-\cos(n+1)\theta_0]\frac{d\theta_0}{\cos\theta_0-\cos\theta}$$

Using Glauert's integral reduces to

$$\frac{-A_n}{2\pi}\pi\left[\frac{\sin(n-1)\theta}{\sin\theta}-\frac{\sin(n+1)\theta}{\sin\theta}\right]=\frac{-A_n}{2}\left[-2\frac{\cancel{\sin\theta}\cos(n\theta)}{\cancel{\sin\theta}}\right]=A_n\cos(n\theta) \quad 3$$

3 1 2 $\Rightarrow$

$$-A_0+\sum_{n=1}^{\infty}A_n\cos(n\theta)=\frac{d\eta_c(\theta)}{dx}-\alpha \qquad (5.50)$$

This is actually a Fourier expansion of the right-hand side of the equation that includes the information on the airfoil geometry

5.3 Classical Solution of the Lifting Problem

Multiplying both sides of the equation (5.50) by cosmθ

$$\cos m\theta \left(-A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) = \frac{d\eta_c(\theta)}{dx} - \alpha \right)$$

&

performing an integration from $0 \rightarrow \pi$

for each value of n, will result in the cancellation of all the nonorthogonal multipliers (where $m \neq n$).

Consequently, for each value of n the value of corresponding coefficient A_n is obtained

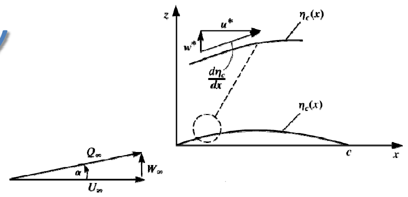
$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} \frac{d\eta_c(\theta)}{dx}\, d\theta, \quad n = 0 \tag{5.51}$$

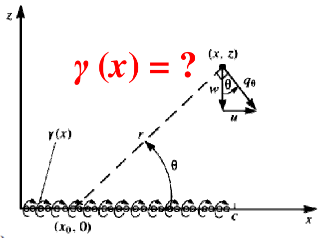
$$A_n = \frac{2}{\pi}\int_0^{\pi} \frac{d\eta_c(\theta)}{dx} \cos n\theta \, d\theta, \quad n = 1, 2, 3, \ldots \tag{5.52}$$

For a given airfoil geometry, the mean camberline $\eta_c(x)$ is a known function and the coefficients A_0, A_1, A_2, . . . can be computed by Eqs. (5.51) & (5.52).

5.3 Classical Solution of the Lifting Problem

Summery





$$\frac{-1}{2\pi}\int_0^{c} \gamma(x_0)\frac{dx_0}{x - x_0} = Q_\infty\left(\frac{d\eta_c}{dx} - \alpha\right), \quad 0 < x < c$$

$$\gamma(x = c) = 0$$

transformation

$$\begin{cases} x = \dfrac{c}{2}(1 - \cos\theta) \\ dx = \dfrac{c}{2}\sin\theta\, d\theta \end{cases}$$

$$\gamma(\theta) = 2Q_\infty\left[A_0\frac{1 + \cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta)\right]$$

Suggested

$$\boxed{\frac{-1}{2\pi}\int_0^{\pi} 2Q_\infty\left[A_0\frac{1 + \cos\theta_0}{\sin\theta_0} + \sum_{n=1}^{\infty} A_n \sin(n\theta_0)\right]\frac{\sin\theta_0\, d\theta_0}{\cos\theta_0 - \cos\theta} = Q_\infty\left[\frac{d\eta_c(\theta)}{dx} - \alpha\right]}$$

$$-A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) = \frac{d\eta_c(\theta)}{dx} - \alpha$$

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} \frac{d\eta_c(\theta)}{dx}\, d\theta, \quad n = 0$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \frac{d\eta_c(\theta)}{dx} \cos n\theta \, d\theta, \quad n = 1, 2, 3, \ldots$$

5.3 Classical Solution of the Lifting Problem

Note that Eq. (5.50) can be rewritten as an expansion of the downwash distribution $w = w(\theta)$ on the airfoil:

$$\left.\begin{aligned} &\frac{\partial \Phi(x,0)}{\partial z} = w(x,0) = Q_\infty\left(\frac{d\eta_c}{dx} - \alpha\right) \\ &-A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) = \frac{d\eta_c(\theta)}{dx} - \alpha \end{aligned}\right\} \Rightarrow \frac{w}{Q_\infty} = -A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) \quad (5.53)$$

the downwash due to the first term (multiplied by A_0) of the vortex distribution is constant along the airfoil chord

The slope $d\eta_c/dx$ can be expanded as a Fourier series such that:

$$\frac{d\eta_c(\theta)}{dx} = \sum_{n=0}^{\infty} B_n \cos(n\theta)$$

comparison with Eq. (5.50) indicates that

$$B_0 = \alpha - A_0, \qquad B_n = A_n \quad n = 1, 2, \ldots, \infty$$

$$\frac{w}{Q_\infty} = -\alpha + \sum_{n=0}^{\infty} B_n \cos(n\theta) \quad (5.53a) \Leftarrow$$

contributions of the AOA & camber to the downwash explicitly

5.4 Aerodynamic Forces and Moments on a Thin Airfoil

Given airfoil geometry ($\eta_c(x)$)

↓

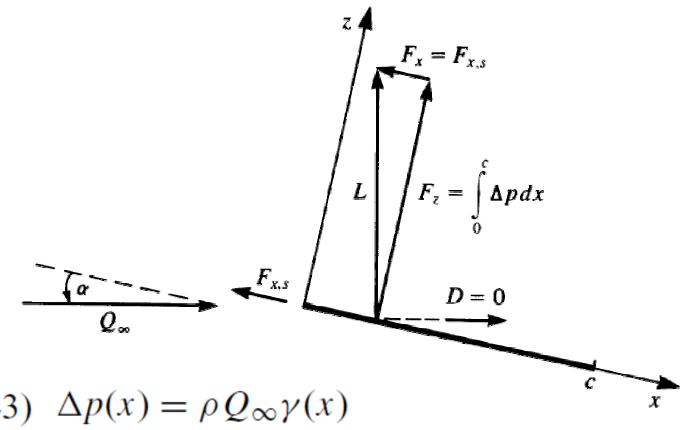
Computing $A_0, A_1, A_2, \ldots$

↓

Calculating pressure difference acroos thin airfoil ← Eq. (5.43) $\Delta p(x) = \rho Q_\infty \gamma(x)$

↓

Evaluating aerodynamic coefficients



Since AOA is small Q_∞ is used instead of $Q_\infty \cos \alpha$. The normal force F_z is then:

$$F_z = \int_0^c \Delta p(x)\,dx = \int_0^c \rho Q_\infty \gamma(x)\,dx = \rho Q_\infty \Gamma$$

where

$$\Gamma = \int_0^c \gamma(x)\,dx \quad (5.54)$$

5.4 Aerodynamic Forces and Moments on a Thin Airfoil

The flat plate of is very thin and the x component of the force is zero

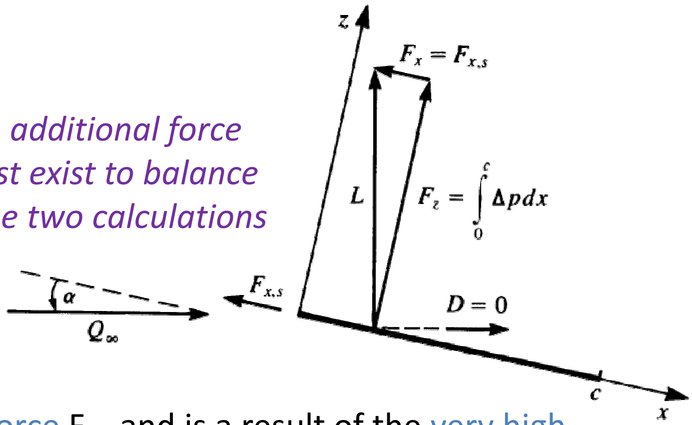
$$F_x = 0$$

$$L = F_z, \qquad D = F_z\alpha$$

From the *Kutta–Joukowski* theorem:

$$L = \rho Q_\infty \Gamma \qquad D = 0$$

an additional force must exist to balance these two calculations



This force is called the leading-edge suction force $F_{x.s}$ and is a result of the very high suction forces acting at the leading edge (where $q \rightarrow \infty$ and the local leading-edge radius is approaching zero). Using the exact solution (Section 6.5.2) near the leading edge of the flat plate & for the small angle of attack case is

$$F_{x.s} = -\rho Q_\infty \Gamma \alpha$$

This force cancels the drag component of the thin lifting airfoil obtained by integrating the pressure difference, so that the two-dimensional drag becomes zero.
This result – that the aerodynamic drag in two-dimensional inviscid incompressible flow is zero. ← *d'Alembert's paradox (1744)*

5.4 Aerodynamic Forces and Moments on a Thin Airfoil

To evaluate the lift of the thin airfoil, the circulation of Eq. (5.54) is calculated

$$\Gamma = \int_0^c \gamma(x)\,dx = \int_0^\pi \gamma(\theta)\frac{c}{2}\sin\theta\,d\theta$$

$$= 2Q_\infty \int_0^\pi \left[A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta)\right]\frac{c}{2}\sin\theta\,d\theta$$

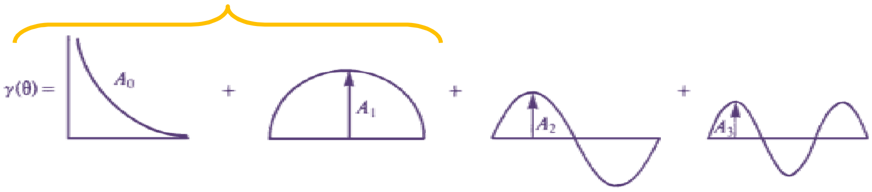
$$\int_0^\pi (1+\cos\theta)\,d\theta = \pi$$

$$\int_0^\pi \sin n\theta \sin\theta\,d\theta = \begin{pmatrix} \frac{\pi}{2} & \text{when} \quad n = 1 \\ 0 & \text{when} \quad n \neq 1 \end{pmatrix}$$

$$\Rightarrow \Gamma = Q_\infty c\pi\left(A_0 + \frac{A_1}{2}\right)$$

$$L = \rho Q_\infty^2 c\pi\left(A_0 + \frac{A_1}{2}\right)$$

indicates that only the first two terms of the circulation will have an effect on the lift and the integration over the airfoil of the higher-order terms will cancel out



5.4 Aerodynamic Forces and Moments on a Thin Airfoil

The pitching moment about the y axis (L.E.)
(positive for a clockwise rotation)

$$M_0 = -\int_0^c \Delta p x \, dx = -\rho Q_\infty \int_0^\pi \gamma(\theta)\frac{c}{2}(1-\cos\theta)\frac{c}{2}\sin\theta \, d\theta$$

$$= \rho Q_\infty \left[-\frac{c}{2}\Gamma + \frac{c^2}{4}\int_0^\pi \gamma(\theta)\sin\theta\cos\theta \, d\theta \right]$$

Some trigonometric manipulations

$$M_0 = -\frac{c}{2}L + \rho\frac{c^2}{4}Q_\infty^2\left(A_0\pi + A_2\frac{\pi}{2}\right) \Rightarrow M_0 = -\rho Q_\infty^2 \pi \frac{c^2}{4}\left(A_0 + A_1 - \frac{A_2}{2}\right)$$

The moment M along the x axis can be described in terms of the lift and the moment at the leading edge as

$$M = M_0 + x \cdot F_z \approx M_0 + x \cdot L$$

The center of pressure x_{cp} is defined as the point where the moment is zero

$$x_{cp} = \frac{-M_0}{L} = \frac{c}{4}\frac{A_0 + A_1 - (A_2/2)}{A_0 + (A_1/2)} \Leftarrow M = M_0 + x_{cp} \cdot L = 0$$

5.4 Aerodynamic Forces and Moments on a Thin Airfoil

The airfoil section aerodynamic coefficients can be derived:

$$C_l = \frac{L}{(1/2)\rho Q_\infty^2 c} = 2\pi\left(A_0 + \frac{A_1}{2}\right) \quad (5.62)$$

$$C_d = \frac{D}{(1/2)\rho Q_\infty^2 c} = 0 \quad (5.63)$$

$$C_{m_0} = \frac{M_0}{(1/2)\rho Q_\infty^2 c^2} = -\frac{\pi}{2}\left[A_0 + A_1 - \frac{A_2}{2}\right] \quad (5.64)$$

Only function of α

$$\begin{cases} A_0 = \alpha - \dfrac{1}{\pi}\displaystyle\int_0^\pi \frac{d\eta_c(\theta)}{dx}\, d\theta, & n = 0 \\ A_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi \frac{d\eta_c(\theta)}{dx}\cos n\theta \, d\theta, & n = 1 \end{cases}$$

$$C_l = 2\pi\left(\alpha - \frac{1}{\pi}\int_0^\pi \frac{d\eta_c(\theta)}{dx}\, d\theta + \frac{A_1}{2}\right) \quad (5.65)$$

For flat plate $d\eta_c/dx = 0$

$$C_l = 2\pi(\alpha - \alpha_{L0}) \qquad \alpha_{L0} = -\frac{1}{\pi}\int_0^\pi \frac{d\eta_c}{dx}(\cos\theta - 1)\, d\theta \quad (5.67)$$

zero-lift angle: a function of the camber

5.4 Aerodynamic Forces and Moments on a Thin Airfoil

$$C_l = 2\pi\left(A_0 + \frac{A_1}{2}\right)$$

$$B_0 = \alpha - A_0,$$

$$B_n = A_n \quad n = 1, 2, \ldots, \infty$$

$$\Rightarrow \quad C_l = 2\pi\left(\alpha - B_0 + \frac{B_1}{2}\right)$$

$$\alpha_{L0} = B_0 - \frac{B_1}{2}$$

$$\frac{d\eta_c(\theta)}{dx} = \sum_{n=0}^{\infty} B_n \cos(n\theta)$$

The lift slope can be defined as:

$$C_{l_\alpha} \equiv \frac{\partial C_l}{\partial \alpha} = 2\pi$$

the lift slope of a 2D airfoil is 2π and that the camber will have an effect similar to an AOA increment Δα but will not change the lift slope

The pitching moment coefficient (Eq. (5.64)) can be rewritten, using the formula for the lift coefficient:

$$C_{m_0} = -\frac{C_l}{4} + \frac{\pi}{4}\underbrace{(A_2 - A_1)}_{\text{independent of } \alpha}$$

independent of α

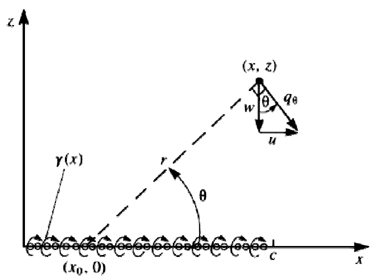
aerodynamic center x_{ac}

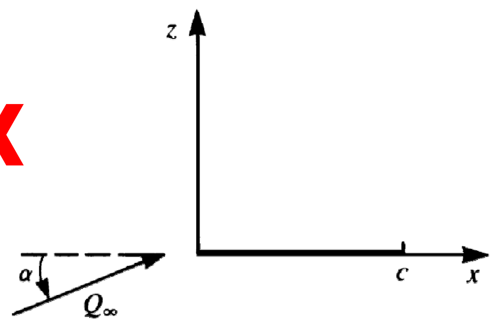
If the moments are calculated relative to the airfoil c/4 point the first term in this equation disappears

$$C_{m_{c/4}} = \frac{\pi}{4}(A_2 - A_1) \quad (5.70)$$

Example 1: Flat Plate

Case I: Thin, lifting model of a symmetric airfoil Represented by a flat plate





$$\eta_c(x) = 0. \quad \Rightarrow \quad A_0 = \alpha, \qquad A_1 = A_2 = \cdots = A_n = 0$$

$$\Gamma = Q_\infty c\pi\left(A_0 + \frac{A_1}{2}\right) \quad \Rightarrow \quad \Gamma = Q_\infty \pi c\alpha$$

$$L = \rho Q_\infty^2 c\pi\left(A_0 + \frac{A_1}{2}\right) \quad \Rightarrow \quad L = \rho Q_\infty \Gamma = \rho Q_\infty^2 \pi c\alpha$$

$$M_0 = -\rho Q_\infty^2 \pi \frac{c^2}{4}\left(A_0 + A_1 - \frac{A_2}{2}\right) \quad \Rightarrow \quad M_0 = -\rho Q_\infty^2 \pi \frac{c^2}{4}\alpha$$

$$\Rightarrow \quad C_l = 2\pi\alpha, \qquad C_{m_0} = -\frac{\pi}{2}\alpha$$

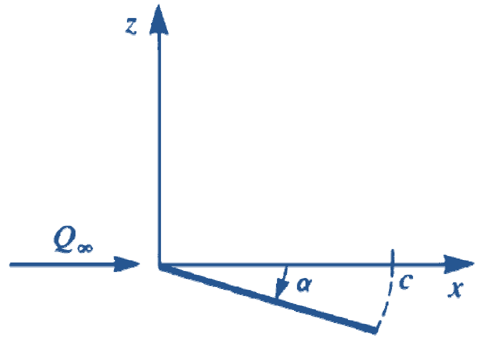
For the symmetric thin airfoil, the center of pressure and the aerodynamic center are Located at the quarter-chord location

center of pressure $\quad \frac{x_{cp}}{c} = \frac{-C_{m_0}}{C_l} = \frac{1}{4}$

Example 1: Flat Plate

Case II: The free-stream angle of attack is zero, but the chord can be expressed as

$$\eta(x) = -\alpha x \quad \Longrightarrow \quad \frac{d\eta}{dx} = -\alpha$$



free-stream coordinate system

$$A_0 = \alpha \quad \text{and} \quad A_1 = A_2 = \cdots = A_n = 0$$

Thus, both methods will lead to the same results.
The pressure coefficient difference, by substituting A_0 & the corresponding circulation

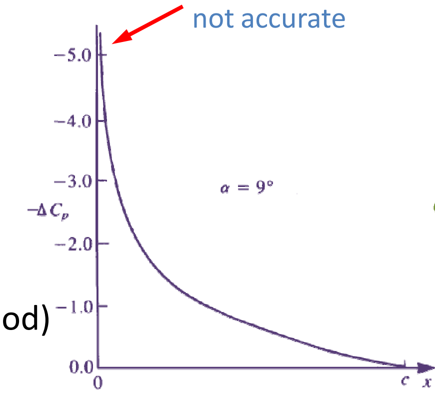
$$\Delta C_p = 2\frac{\gamma}{Q_\infty} = 4\frac{1+\cos\theta}{\sin\theta}\alpha$$

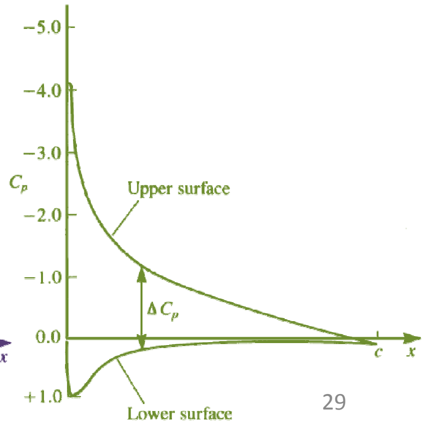
$$\Big\Downarrow \; x = \tfrac{c}{2}(1-\cos\theta)$$

$$\Delta C_p = 4\sqrt{\frac{c-x}{x}}\alpha$$

a comparison with the results of a more accurate method (panel method) for a NACA0012 symmetric airfoil.

Near L.E. the flat plate solution is singular & not accurate



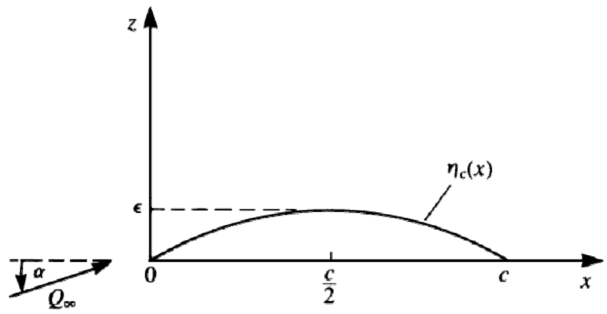


Example 2: Thin Airfoil with a Parabolic Camber

Simple nonsymmetric chordline shape consider the parabolic camberline
ϵ: maximum height

$$\underbrace{\eta_c(x) = 4\epsilon\frac{x}{c}\left[1-\frac{x}{c}\right] \qquad \frac{d\eta_c(x)}{dx} = 4\frac{\epsilon}{c}\left[1-2\frac{x}{c}\right]}$$

$$\Big\Downarrow \; x = \tfrac{c}{2}(1-\cos\theta)$$



$$\frac{d\eta_c(\theta)}{dx} = 4\frac{\epsilon}{c}\left[1-\frac{2}{c}\frac{c}{2}(1-\cos\theta)\right] = 4\frac{\epsilon}{c}\cos\theta$$

Substituting this into Eq. (5.51) & (5.52) $\longrightarrow$

$$\begin{cases} A_0 = \alpha - \dfrac{1}{\pi}\displaystyle\int_0^\pi \frac{d\eta_c(\theta)}{dx}\,d\theta, & n=0 \\ A_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi \frac{d\eta_c(\theta)}{dx}\cos n\theta\,d\theta, & n = 1,2,3,\ldots \end{cases}$$

Because of the orthogonal nature of the integral $\int_0^\pi \cos n\theta \cos m\theta \, d\theta$ all terms where m ≠ n will vanish.

$$A_0 = \alpha - 0 \longleftarrow \textit{for } m = 1$$

And, only the first coefficient will be nonzero

$$A_1 = 4\frac{\epsilon}{c}$$

$$A_2 = A_3 = \cdots = A_n = 0$$

$$\frac{d\eta_c(\theta)}{dx} = \sum_{n=0}^{\infty} B_n\cos(n\theta) = 4\frac{\epsilon}{c}\cos\theta$$

clearly $B_1 = 4\epsilon/c$ & other $B_n = 0$

Example 2: Thin Airfoil with a Parabolic Camber

The lift and the moment of the parabolic camber airfoil:

$$L = \rho Q_\infty^2 \pi c\left(\alpha + 2\frac{\epsilon}{c}\right)$$

$$M_0 = -\rho Q_\infty^2 \pi \frac{c^2}{4}\left(\alpha + 4\frac{\epsilon}{c}\right)$$

$$C_l = 2\pi\left(\alpha + 2\frac{\epsilon}{c}\right)$$

$$C_{m_0} = -\frac{\pi}{2}\left(\alpha + 4\frac{\epsilon}{c}\right)$$

&

$$C_l = 2\pi(\alpha - \alpha_{L0})$$

$$\alpha_{L0} = -2\epsilon/c$$

airfoil will have zero lift when it is pitched to a negative angle of attack with a magnitude of 2ϵ/c

The center of pressure is obtained by dividing the moment by the lift

$$\frac{x_{cp}}{c} = \frac{1}{4}\frac{\alpha + 4\epsilon/c}{\alpha + 2\epsilon/c}$$

Note: at $\alpha = 0$ the center of pressure is at the c/2 & as AOA increases it moves toward the c/4.

The pitching moment about the aerodynamic center from Eq. (5.70):

$$C_{m_{c/4}} = \frac{\pi}{4}(A_2 - A_1) = -\pi\frac{\epsilon}{c}$$

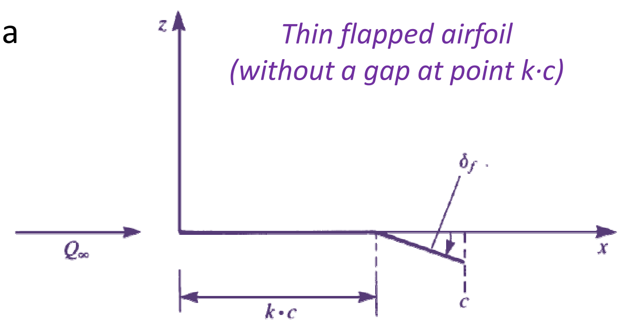
which indicates that the portion of the moment that is independent of AOA increases with increased curvature (as ϵ/c increases) of the camberline.

Example 3: Flapped Airfoil

The main airfoil plane is placed on the *x* axis, & at a chordwise position *k* · *c* the flap is deflected by *δf* for *α* = 0

$$\frac{d\eta_c}{dx} = 0 \qquad \text{for} \quad 0 < x < kc$$

$$\frac{d\eta_c}{dx} = -\delta_f \qquad \text{for} \quad kc < x < c$$



One of the most frequently used control devices is the trailing-edge flap. The reason for mounting such a device at T.E. can be observed by examining the ($\cos\theta - 1$) term in Eq. (5.67).

$$\alpha_{L0} = -\frac{1}{\pi}\int_0^\pi \frac{d\eta_c}{dx}\underbrace{(\cos\theta - 1)}\,d\theta$$

This implies that the zero-lift angle is most influenced by the T.E. region where $\theta \to \pi$; therefore, relatively small deflections of the flap at the T.E. will have noticeable effect.

Since the coefficients A_n are given as a function of the variable θ, the location of the hinge point θ_k can be found by using $x = \frac{c}{2}(1 - \cos\theta)$

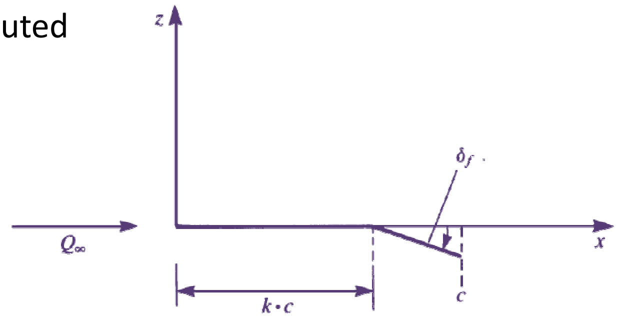
$$kc = \frac{c}{2}(1 - \cos\theta_k) \quad \Longrightarrow \quad \cos\theta_k = 1 - 2k$$

Example 3: Flapped Airfoil

The coefficients of Eqs. (5.51) and (5.52) are computed

$$A_0 = \alpha + \frac{1}{\pi}\int_{\theta_k}^{\pi} \delta_f \, d\theta = \alpha + \frac{\delta_f}{\pi}(\pi - \theta_k)$$

$$A_n = -\frac{2}{\pi}\int_{\theta_k}^{\pi} \delta_f \cos n\theta \, d\theta = \frac{2\delta_f}{\pi}\frac{\sin n\theta_k}{n}$$



The lift and pitching moment coefficients

$$C_l = 2\pi\left\{\alpha + \delta_f\left[\left(1 - \frac{\theta_k}{\pi}\right) + \frac{1}{\pi}\sin\theta_k\right]\right\}$$

$$C_{m_0} = -\frac{\pi}{2}\left[\alpha + \delta_f\left(1 - \frac{\theta_k}{\pi}\right) + \frac{2\delta_f}{\pi}\sin\theta_k - \frac{\delta_f}{2\pi}\sin 2\theta_k\right]$$

Setting $\alpha = 0$ allows the incremental effect of the flap to be obtained

$$\Delta C_l = [2(\pi - \theta_k) + 2\sin\theta_k]\,\delta_f$$

$$\Delta C_{m_0} = -\frac{1}{2}\left[(\pi - \theta_k) + 2\sin\theta_k - \frac{1}{2}\sin 2\theta_k\right]\delta_f$$

The increment in the moment at the aerodynamic center, *c*/4, due to the flap deflection is obtained using Eq. (5.70) as

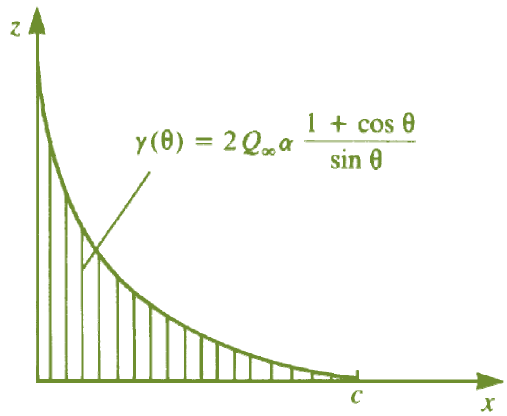
$$\Delta C_{m_{c/4}} = \left[\frac{1}{4}\sin 2\theta_k - \frac{1}{2}\sin\theta_k\right]\delta_f$$

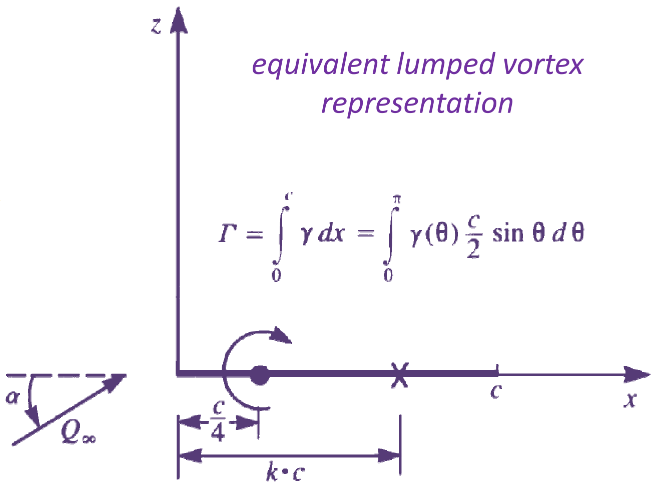
5.5 The Lumped-Vortex Element

Developing a simple lifting element based on the results for the lifting symmetrical airfoil (flat plate)

$$\gamma(\theta) = 2Q_\infty\alpha\frac{1 + \cos\theta}{\sin\theta} \quad (5.97)$$

vortex distribution on a flat plate airfoil





The vortex distribution can be replaced by a single vortex with:

1- The same strength $\Gamma = \int_0^c \gamma(x)\,dx$

2- Since the lift of the symmetric airfoil $L = \rho Q_\infty \Gamma$ acts at the center of pressure (at c/4), the concentrated vortex is placed there.

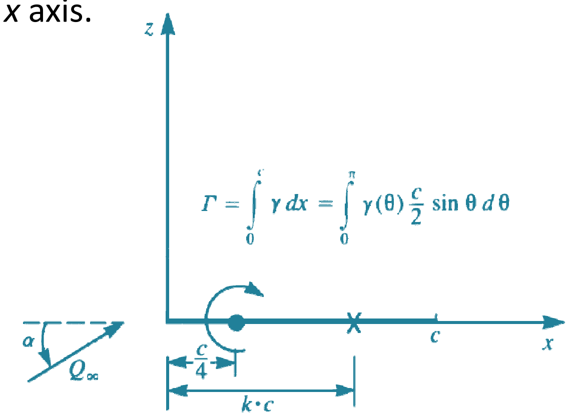
5.5 The Lumped-Vortex Element

Representing lifting flat plate by only one vortex Γ

Specifying B.C. zero normal flow at the surface at only one point

Assuming that this point is at a distance $k \cdot c$ along the x axis. specifying B.C. of zero normal velocity as:

$$\frac{-\Gamma}{2\pi[kc-(1/4)c]} + Q_\infty \alpha = 0 \qquad (5.98)$$



for this model

$$\Gamma = \pi c Q_\infty \alpha$$

circulation for a flat plate Example 1

$$\frac{-\pi c Q_\infty \alpha}{2\pi[kc-(1/4)c]} + Q_\infty \alpha = 0$$

The point at which B.C. needs to be specified (*collocation point*):

$$k = \frac{3}{4}$$

From Generalized Kutta–Joukowski theorem (Chapter 6), the lift force on an airfoil is:

$$L = \rho Q_\infty \Gamma \left(1 + \frac{\mathbf{Q}_\infty \cdot \mathbf{q}_I}{Q_\infty^2}\right) \qquad (5.100)$$

$\mathbf{q}_I$ is the velocity induced by other vortices at the airfoil vortex location

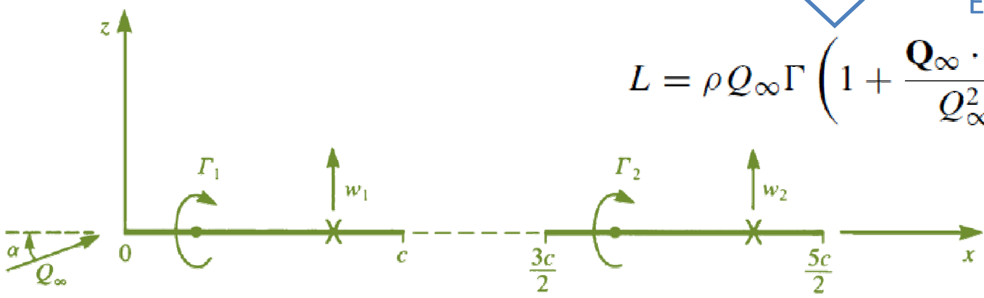
Example 1: Tandem Airfoils

The lift of the two-airfoil system, Γ_1 and Γ_2 are the circulations of the two airfoils
The two B.C. at the two collocation points require that the normal velocity component will be zero (Influence of the two vortices + free-stream normal component):

$$\left.\begin{aligned} w_1 &= \frac{-\Gamma_1}{2\pi c/2} + \frac{\Gamma_2}{2\pi c} + Q_\infty \alpha = 0 \\ w_2 &= \frac{-\Gamma_1}{2\pi 2c} + \frac{-\Gamma_2}{2\pi c/2} + Q_\infty \alpha = 0 \end{aligned}\right\} \Rightarrow \Gamma_1 = \frac{4}{3}\pi c Q_\infty \alpha, \qquad \Gamma_2 = \frac{2}{3}\pi c Q_\infty \alpha$$

force on each airfoil Eq. (5.100)

$$L = \rho Q_\infty \Gamma \left(1 + \frac{\mathbf{Q}_\infty \cdot \mathbf{q}_I}{Q_\infty^2}\right)$$



The front airfoil has a larger lift owing to the upwash induced by the second airfoil, and because of the same but reversed interaction the second airfoil will have less lift. Also, this effect is stronger when the airfoils are closer and the interaction will disappear as the distance increases.

Note: The immediate effects of the tandem airfoil configuration could be estimated with minimum effort.

Example 2: Ground Effect

The airfoil near the ground, which is modeled by using the mirror-image method. to create a straight streamline at the ground plane two symmetrically positioned airfoils are considered.

B.C. at the collocation point, using lumped-vortex element:

$$-\frac{\Gamma}{\pi c} + \mathbf{q}_I \cdot \mathbf{n} + Q_\infty \sin\alpha = 0$$

Collection point at $\begin{cases} x = (c/2)\cos\alpha \\ z = -(c/2)\sin\alpha \end{cases}$

image vortex at $\begin{cases} x_0 = 0 \\ z_0 = -2h \end{cases}$ ⇒ $(u, w) = \frac{\Gamma}{2\pi}\frac{(z - z_0, x_0 - x)}{(x - x_0)^2 + (z - z_0)^2}$ velocity due to image vortex

Note: The circulation of the image vortex is (−Γ)

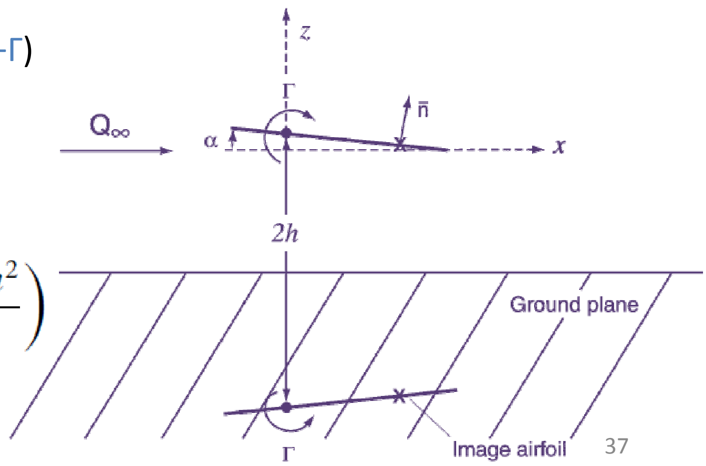
The normal to the airfoil

$$\mathbf{n} = \sin\alpha\,\mathbf{i} + \cos\alpha\,\mathbf{k}$$

Resulting circulation:

$$\Gamma = \underbrace{\pi Q_\infty c \sin\alpha}_{\text{exact solution for the flat plate in the absence of the ground plane}} \left(\frac{1 - (c/2h)\sin\alpha + c^2/16h^2}{1 - (c/4h)\sin\alpha}\right)$$

exact solution for the flat plate in the absence of the ground plane



Example 2: Ground Effect

The lift force on the airfoil (using Eq. (5.100)):

$$L = \rho Q_\infty \Gamma\left(1 - \frac{\Gamma}{4\pi Q_\infty h}\right) \leftarrow \Gamma = \pi Q_\infty c \sin\alpha \left(\frac{1 - (c/2h)\sin\alpha + c^2/16h^2}{1 - (c/4h)\sin\alpha}\right)$$

c/h → 0
airfoil far from the ground

$$L = \pi\rho Q_\infty^2 c\ \sin\alpha\left[1 - \frac{c}{2h}\sin\alpha + \frac{c^2}{16h^2}(1 + \sin^2\alpha) + O\left(\frac{c^3}{h^3}\right)\right]$$

Corresponding results for a parabolic arc airfoil $\eta_c(x) = 4\epsilon\frac{x}{c}\left[1 - \frac{x}{c}\right]$ at zero AOA in ground effect:

$$\Gamma = 2\pi Q_\infty \epsilon\left(\frac{1 + c^2/16h^2}{1 + \epsilon/2h}\right)$$

Where ϵ is the maximum camber & h is measured from midchord.

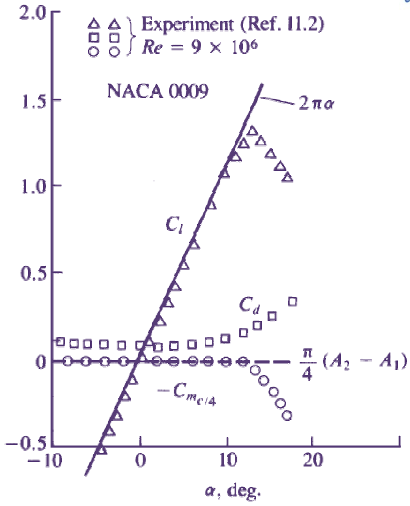
The lift force for large ground height

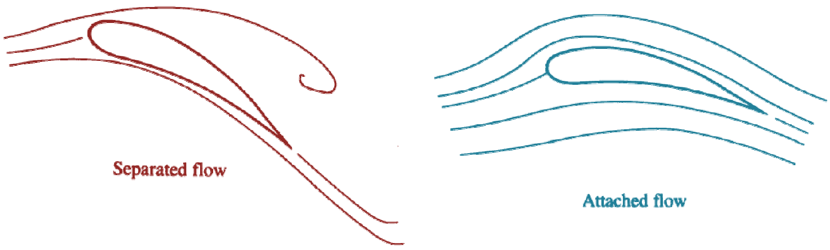
$$L = 2\pi\rho Q_\infty^2 \epsilon\left[1 - \frac{\epsilon}{h} + \frac{c^2}{16h^2} + \frac{3}{2}\frac{\epsilon^2}{h^2} + O\left(\frac{1}{h^3}\right)\right]$$

5.6 Summary and Conclusions from Thin Airfoil Theory

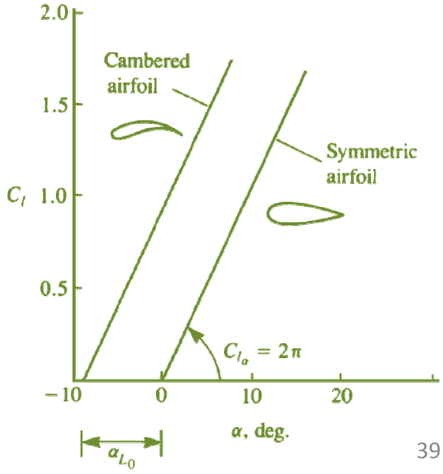
1. The lift slope of a 2D airfoil is 2π, (Eq. 5.66).

2. The pitching moment at the aerodynamic center (at c/4) is independent of AOA. (excluding at stall).





3. Airfoil camber does not change the lift slope and can be viewed as an additional AOA effect (α_{L0} in Eq. (5.66)). The symmetric airfoil will have zero lift at $\alpha = 0$ while the airfoil with camber has an "effective" AOA that is larger by α_{L0}



5.6 Summary and Conclusions from Thin Airfoil Theory

4. The T.E. section has a larger influence on the camber effect. Therefore, if the lift of the airfoil needs to be changed without changing its AOA, then changing the chordline geometry (e.g., by flaps or slats) at the T.E. region is more effective than at the L.E. region.

5. The effect of thickness on the airfoil lift is not treated in a satisfactory manner by the small-disturbance approach, but this will be calculated more accurately in the chapters 6 & 7.

6. The 2D drag coefficient obtained by this model is zero and there is no drag associated with the generation of 2D lift. Experimental airfoil data, however, include drag due to the viscous boundary layer on the airfoil, and this should be included in engineering calculations. The experimental drag coefficient values for the NACA 0009 airfoil at the "zero-lift" drag coefficient is close to $C_d = 0.0055$